

Derivation of Euler-Lagrange Equation for ROF denoising algorithm

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1 Rudin-Osher-Fatemi TV-L2 Denoising Model

Since this article is not meant for comprehensive introduction to the problem, I will not thoroughly review theoretic cues. Depending on which viewpoint one takes upon the problem, a common framework of energy formulation on desirable denoised image u given a noisy picture u_0 is

$$E[u|u_0] = \lambda \|u - u_0\|_2^2 + \|\nabla u\|_1 \quad (1)$$

where the first term measures closeness of desirable solution to the noisy picture and the second is smoothness penalty - also known as regularization - that penalizes non-smoothness on pursued solution. λ , which originates from Lagrange multiplier, plays a *balancing* role between noise model and image's smoothness requirement. In this formulation, large λ value weighs closeness and small one focuses on smoothing effect. First-order condition for a minimizer u^* is that its *derivative* $\nabla E = 0$ and this Gâteaux derivative is achieved via Euler-Lagrange equation. A classical solution is to proceed with time-marching method,

$$u_t = -\nabla E[u|u_0] \quad (2)$$

to reach at minimum energy level. For reference, see Rudin, Osher, Fatemi (1992) " *Nonlinear total variation based image removal algorithm*" [1].

2 Basic Calculus

First we need is *divergence theorem*. Suppose Ω is a compact subset of \mathbb{R}^n with a piecewise smooth boundary denoted by $\partial\Omega$. If \mathbf{F} is a continuously differentiable vector field which is defined on Ω , we have

$$\int_{\Omega} (\nabla \cdot \mathbf{F}) dV = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) dS, \quad (3)$$

where left part is volume integral over Ω which expresses, in some sense, change within the entire volume and right side is surface integral over the boundary $\partial\Omega$ of Ω , which somehow measures outwards flux on its surface.

In addition, we need following equality

$$\int_{\Omega} \nabla \cdot (u \nabla v) = \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} u \nabla \cdot (\nabla v) = \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} u (\nabla^2 v). \quad (4)$$

3 Derivation of Euler-Lagrange equation

A standard Euler-Lagrange equation pertains to the concept of directional derivative in that we can achieve in via following

$$0 = \nabla E = \left. \frac{d}{dt} \right|_{t=0} E[u + tv] \quad (5)$$

and calculation is as follows.

$$\begin{aligned} \frac{d}{dt} E[u + tv] &= \frac{d}{dt} \left(\lambda \int_{\Omega} (u + tv - u_0)^2 + \int_{\Omega} |\nabla(u + tv)| \right) \\ &= \lambda \frac{d}{dt} \left(\int_{\Omega} u^2 + t^2 v^2 + u_0^2 + 2tuv - 2tu_0v - 2uu_0 \right) \\ &\quad + \frac{d}{dt} \left(\int_{\Omega} |\nabla(u + tv)| \right) \\ &= \lambda \int_{\Omega} (2tv^2 + 2uv - 2u_0v) + \frac{d}{dt} \left(\int_{\Omega} |\nabla(u + tv)| \right). \end{aligned}$$

For the second term above, using chain rule, we have

$$\frac{d}{dt} \left(\int_{\Omega} |\nabla(u + tv)| \right) = \int_{\Omega} \frac{\nabla(u + tv)}{|\nabla(u + tv)|} \cdot \nabla v$$

since

$$\frac{d}{dx} |x| = \frac{x}{|x|} = \text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ (-1, 1) & \text{otherwise.} \end{cases}$$

Combining these two and inserting $t = 0$, we have

$$0 = \nabla E = 2\lambda \int_{\Omega} ((u - u_0)v) + \int_{\Omega} \left(\frac{\nabla u}{|\nabla u|} \cdot \nabla v \right) \quad (6)$$

and we need to further apply basic calculus results from the above in that

$$\begin{aligned} \int_{\Omega} \left(\frac{\nabla u}{|\nabla u|} \cdot \nabla v \right) &= - \int_{\Omega} \left(\nabla \cdot \frac{\nabla u}{|\nabla u|} \right) v + \int_{\Omega} \nabla \cdot v \frac{\nabla u}{|\nabla u|} \\ &= - \int_{\Omega} \text{div} \left(\frac{\nabla u}{|\nabla u|} \right) v + \int_{\Omega} \text{div} \left(v \frac{\nabla u}{|\nabla u|} \right) \\ &= - \int_{\Omega} \text{div} \left(\frac{\nabla u}{|\nabla u|} \right) v + \int_{\partial\Omega} \frac{v}{|\nabla u|} (\nabla u \cdot n). \end{aligned}$$

Therefore, if we set $\nabla u \cdot n = 0$, we have

$$0 = \int_{\Omega} \left((2\lambda(u - u_0)) - \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) \right) v \quad (7)$$

and since it should be satisfied with arbitrary direction v , we finally have

$$\begin{cases} (u - u_0) - \frac{1}{2\lambda} \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (8)$$

Then, time-marching methods can be written as following differential equation with Neumann boundary condition,

$$\begin{cases} \frac{\partial u}{\partial t} = (u_0 - u) + \frac{1}{2\lambda} \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

References

- [1] Leonid I. Rudin, Stanley Osher, and Emad Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D: Nonlinear Phenomena*, 60(1-4):259–268, November 1992.